

# On W. Gordon's integral (1929) and related identities

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Analytic evaluation of Gordon's integral

$$J_c^{j(\pm p)}(b, b'; \lambda, w, z) = \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c \pm p; zx) dx,$$

are given along with convergence conditions. It shows enormous number of definite integrals, frequently appear in theoretical and mathematical physics applications, easily deduced from this generalized integral.

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## I. GORDON'S INTEGRAL: INTRODUCTION

Among the important integrals in theoretical and mathematical physics is W. Gordon's integral [3], see also [4, 5, 8, 10],

$$J_c^{j(\pm p)}(b, b'; \lambda, w, z) = \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c \pm p; zx) dx$$

$$(c + j > 0; \lambda > 0; c, c \pm p \neq 0, -1, -2, \dots; p \geq 0; j = 0, \pm 1, \pm 2, \dots), \quad (1)$$

where  ${}_1F_1$  is the confluent hypergeometric function  ${}_1F_1(b; c; z) = \sum_{k=0}^\infty (b)_k z^k / [(c)_k k!]$  in which  $(b)_k = b(b+1) \dots (b+n-1) = \Gamma(b+k)/\Gamma(b)$  is the Pochhammer symbol defined in terms of Gamma function. The massive uses of this integral and the subclasses of it span large volume of research papers and monographs [4–6, 8, 10]. It was proven (Lemma 1 in [8]) that, for  $c + j > 0$  and  $|w| + |z| < |\lambda|$ ,

$$J_c^{j(\pm p)}(b, b'; \lambda, w, z) = \frac{\Gamma(c+j)}{\lambda^{c+j}} F_2 \left( c+j; \begin{matrix} b, & b' \\ c, & c \pm p \end{matrix}; \frac{w}{\lambda}, \frac{z}{\lambda} \right), \quad (2)$$

where the second Appell function reads ([1], equation (2))

$$F_2 \left( \begin{matrix} a; b, b' \\ c, c' \end{matrix}; w, z \right) \equiv F_2(a; b, b'; c, c'; w, z) = \sum_{m=0}^\infty \sum_{p=0}^\infty \frac{(a)_{m+p} (b)_m (b')_p}{(c)_m (c')_p} \frac{w^m z^p}{m! p!}, \quad (c, c' \neq 0, -1, \dots; |w| + |z| < 1). \quad (3)$$

Exact analytical expressions of this integral by means of more elementary functions are given in the present work, where many subclasses are analysed and evaluated in simplified expressions allow for faster computations.

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## II. GORDON'S INTEGRAL: CLOSED FORM EXPRESSIONS

By means of the double integral representation of the second Appell function ([1], equation 7; see also [8]), for  $c, c-p \neq 0, -1, -2, \dots, j, p = 0, 1, 2, \dots, |w| + |z| < 1$ , it follows, for  $j \geq p$ , that

$$\begin{aligned} J_c^{j(\pm p)}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c \pm p; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j-b'}(\lambda-z)^{b'}} \sum_{k=0}^{j \mp p} \frac{(-j \pm p)_k (b')_k}{(c \pm p)_k k!} \left(1 - \frac{\lambda}{z}\right)^{-k} F_1\left(b, c+j-b', b'+k; c; \frac{w}{\lambda}, \frac{w}{\lambda-z}\right), \\ &\quad (c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p \geq 0; j = 0, \pm 1, \dots; |w| + |z| < \lambda), \end{aligned} \quad (4)$$

particularly, for  $p = j = 0, 1, 2, \dots$ ,

$$\begin{aligned} J_c^{jj}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c+j; zx) dx = \frac{\Gamma(c+j) F_1\left(b, c+j-b', b'; c; \frac{w}{\lambda}, \frac{w}{\lambda-z}\right)}{\lambda^{c+j-b'}(\lambda-z)^{b'}}, \\ &\quad (c+j > 0; \lambda > 0; c \neq 0, -1, \dots; j = 0, \pm 1, \dots; |w| + |z| < \lambda), \\ J_c^{jj}(c+j, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c+j; c; wx) {}_1F_1(b'; c+j; zx) dx \\ &= \frac{\Gamma(c+j)(\lambda-w)^{b'-c-j}}{(\lambda-z-w)^{b'}} F_1\left(-j, c+j-b', b'; c; \frac{w}{w-\lambda}, \frac{w}{w+z-\lambda}\right), \\ &\quad (c+j > 0; \lambda > 0; c \neq 0, -1, \dots; j = 0, \pm 1, \dots; |w| + |z| < \lambda), \\ J_c^{jj}(c+j, c; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c+j; c; wx) {}_1F_1(c; c+j; zx) dx \\ &= \frac{\Gamma(c+j)}{(\lambda-w)^j(\lambda-z-w)^c} F_1\left(-j, j, c; c; \frac{w}{w-\lambda}, \frac{w}{w+z-\lambda}\right), \\ &\quad (c+j > 0; \lambda > 0; c \neq 0, -1, \dots; j = 0, \pm 1, \dots; |w| + |z| < \lambda), \end{aligned} \quad (5)$$

where  $F_1$  is the first Appell function ([1], equation (1)). By mean of ([9], formula (8.3.5))

$$F_1(a; b, b'; c; w, z) = (1-w)^{-a} F_1\left(a, c-b-b', b'; c; \frac{w}{w-1}, \frac{z-w}{1-w}\right), \quad (6)$$

it follows, for  $j \geq p$ ,

$$\begin{aligned} J_c^{j(\pm p)}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c \pm p; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j-b-b'}(\lambda-w)^b(\lambda-z)^{b'}} \sum_{k=0}^{j \mp p} \frac{(-j \pm p)_k (b')_k}{(c \pm p)_k k!} \left(1 - \frac{\lambda}{z}\right)^{-k} \\ &\quad \times \sum_{r=0}^{j+k} \frac{(b)_r (-j-k)_r}{(c)_r r!} \left(1 - \frac{\lambda}{w}\right)^{-r} {}_2F_1\left(b+r, b'+k; c+r; \frac{wz}{(\lambda-z)(\lambda-w)}\right), \\ &\quad (c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, -2, \dots; |w| + |z| < \lambda), \end{aligned} \quad (7)$$

where for  $p = j = 0, 1, 2, \dots$

$$\begin{aligned} J_c^{jj}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c+j; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j-b-b'}(\lambda-w)^b(\lambda-z)^{b'}} \sum_{r=0}^j \frac{(b)_r (-j)_r}{(c)_r r!} \left(1 - \frac{\lambda}{w}\right)^{-r} {}_2F_1\left(b+r, b'; c+r; \frac{wz}{(\lambda-z)(\lambda-w)}\right), \\ &\quad (c+j > 0; \lambda > 0; c, c+j \neq 0, -1, -2, \dots; |w| + |z| < \lambda). \end{aligned} \quad (8)$$

Setting  $b' = c + j$ , equation (4) yield

$$\begin{aligned} J_c^{j(\pm p)}(b, c + j; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(c + j; c \pm p; zx) dx \\ &= \frac{\Gamma(c + j)}{(\lambda - z)^{c+j-b} (\lambda - z - w)^b} \sum_{k=0}^{j \mp p} \frac{(-j \pm p)_k (c + j)_k}{(c \pm p)_k k!} \left(1 - \frac{\lambda}{z}\right)^{-k} {}_2F_1\left(b, -j - k; c; \frac{w}{w + z - \lambda}\right), \\ &\quad (c + j > 0; \lambda > 0; c, c \pm p \neq 0, -1, -2, \dots; p \geq 0; j = 0, \pm 1, \pm 2, \dots; |w| + |z| < \lambda), \end{aligned} \quad (9)$$

By means of the Kummer's first transformation  ${}_1F_1(b; c; z) = e^z {}_1F_1(c - b; c; -z)$ , and the series representation

$$F_2(a; b, b'; c, c'; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m {}_2F_1(a + m, b'; c', y), \quad (10)$$

it easily follows

$$\begin{aligned} J_c^{j(\pm p)}(c + j, b; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c + j; c; wx) {}_1F_1(b; c \pm p; zx) dx \\ &= \frac{\Gamma(c + j)}{(\lambda - w)^{c+j} (\lambda - w - z)^b} \sum_{k=0}^j \frac{(-j)_k (c + j)_k}{(c)_k k!} \left(1 - \frac{\lambda}{w}\right)^{-k} {}_2F_1\left(b, c + j + k; c \pm p; \frac{z}{\lambda - w}\right), \\ &\quad (c + j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; |w| + |z| < \lambda). \end{aligned} \quad (11)$$

By means of the identity ([2], formula 5.14.3)

$$\sum_{k=0}^n \binom{n}{k} \frac{(-z)^k}{(b)_k} {}_1F_1(a; b + k; z) = {}_1F_1(a - n; b; z), \quad (12)$$

it follows that

$$\begin{aligned} J_c^{j(\pm p)}(c - j, b; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c - j; c; wx) {}_1F_1(b; c \pm p; zx) dx \\ &= \frac{\Gamma(c + j)}{\lambda^{c+j}} \sum_{k=0}^j \frac{(-j)_k (c + j)_k}{(c)_k k!} \left(\frac{w}{\lambda}\right)^k F_2\left(c + j + k, c, b; c + k, c \pm p; \frac{w}{\lambda}, \frac{z}{\lambda}\right), \\ &\quad (c + j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; |w| + |z| < \lambda). \end{aligned} \quad (13)$$

By means of the identity ([2], formula 5.14.1)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(b - a)_k}{(b)_k} {}_1F_1(a; b + k; z) = \frac{(a)_n}{(b)_n} {}_1F_1(a + n; b + n; z), \quad (14)$$

it follows that

$$\begin{aligned} J_{c+n}^{(j-n)(\pm p-n)}(b + n, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b + n; c + n; wx) {}_1F_1(b'; c \pm p; zx) dx \\ &= \frac{\Gamma(c + j)}{\lambda^{c+j}} \frac{(c)_n}{(b)_n} \sum_{k=0}^n \frac{(-n)_k (c - b)_k}{(c)_k k!} F_2\left(c + j; b, b'; c + k, c \pm p; \frac{w}{\lambda}, \frac{z}{\lambda}\right), \\ &\quad (c + j > 0; \lambda > 0; n = 0, 1, \dots; c + n, c \pm p \neq 0, -1, \dots; |w| + |z| < \lambda). \end{aligned} \quad (15)$$

By means of the identity ([2], formula 5.14.5)

$${}_1F_1(b + n; c + n; w) = \frac{(c - 1)_n (c)_n}{(b)_n (-w)^n} \sum_{k=0}^n \frac{(-n)_k (1 - c)_k}{(2 - c - n)_k k!} {}_1F_1(b, c - k, w) \quad (16)$$

it follows that

$$\begin{aligned} J_{c+n}^{(j-n)(\pm p-n)}(b+n, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b+n; c+n; wx) {}_1F_1(b'; c \pm p; zx) dx \\ &= \frac{(c-1)_n (c)_n}{(-w)^n (b)_n} \frac{\Gamma(c+j-n)}{\lambda^{c+j-n}} \sum_{k=0}^n \frac{(-n)_k (1-c)_k}{(2-c-n)_k k!} F_2 \left( c+j-n; \begin{matrix} b, & b' \\ c-k, & c \pm p \end{matrix}; \frac{w}{\lambda}, \frac{z}{\lambda} \right), \\ &(c \neq 0, \pm 1; \dots, c+j > 0; \lambda > 0; n = 0, 1, \dots; c+n, c \pm p \neq 0, -1, \dots; |w| + |z| < \lambda). \end{aligned} \quad (17)$$

By means of the identity ([2], formula 5.14.6)

$${}_1F_1(b+n; c; w) = \frac{(b-c+1)_n}{(b)_n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(1-c)_k}{(b-c+1)_k} {}_1F_1(b, c-k, w) \quad (18)$$

it follows

$$\begin{aligned} J_c^{j(\pm p)}(b+n, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b+n; c; wx) {}_1F_1(b'; c \pm p; zx) dx \\ &= \frac{(b-c+1)_n}{(b)_n} \frac{\Gamma(c+j)}{\lambda^{c+j}} \sum_{k=0}^n \frac{(-n)_k (1-c)_k}{(b-c+1)_k k!} F_2 \left( c+j; \begin{matrix} b, & b' \\ c-k, & c \pm p \end{matrix}; \frac{w}{\lambda}, \frac{z}{\lambda} \right) \\ &(c \neq 0, \pm 1, \dots, c+j > 0; \lambda > 0; n = 0, 1, \dots; c, c \pm p \neq 0, -1, \dots; |w| + |z| < \lambda). \end{aligned} \quad (19)$$

By means of the identity ([2], formula 5.14.7)

$${}_1F_1(b-n; c-n; w) = \frac{(w)^n}{(1-c)_n} \sum_{k=0}^n \binom{n}{k} \frac{(1-c)_k}{w^k} {}_1F_1(b, c-k, w) \quad (20)$$

it follows

$$\begin{aligned} J_{c-n}^{(j+n)(\pm p+n)}(b-n, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b-n; c-n; wx) {}_1F_1(b'; c \pm p; zx) dx \\ &= \frac{w^n \Gamma(c+j+n)}{\lambda^{c+j+n} (1-c)_n} \sum_{k=0}^n \frac{(-n)_k (1-c)_k}{k! (1-c-j-n)_k} \left( \frac{\lambda}{w} \right)^k F_2 \left( c+j-k+n; \begin{matrix} b, & b' \\ c-k, & c \pm p \end{matrix}; \frac{w}{\lambda}, \frac{z}{\lambda} \right) \\ &(c \neq 0, \pm 1, \dots, c+j > 0; \lambda > 0; n = 0, 1, \dots; c-n, c \pm p \neq 0, -1, \dots; |w| + |z| < \lambda). \end{aligned} \quad (21)$$

Since ([7], formula 7.2.4.68)

$$F_2(a; b, b; c, c; z, -z) = {}_4F_3 \left( \frac{a}{2}, \frac{a+1}{2}, b, c-b; \frac{c}{2}, \frac{c+2}{2}, c; z^2 \right). \quad (22)$$

it follows that

$$\begin{aligned} J_c^{j0}(b, b; \lambda, w, -w) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b; c; -wx) dx = \frac{\Gamma(c+j)}{\lambda^{c+j}} {}_4F_3 \left( \begin{matrix} b, & c-b, & \frac{c+j}{2}, & \frac{c+j+1}{2} \\ c, & \frac{c}{2}, & \frac{c+1}{2} \end{matrix}; \frac{w^2}{\lambda^2} \right), \\ &(c+j > 0; \lambda > 0; c \neq 0, -1, -2, \dots, |w| < \lambda), \end{aligned} \quad (23)$$

$$\begin{aligned} J_c^{10}(b, b; \lambda, w, -w) &= \int_0^\infty x^c e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b; c; -wx) dx = \frac{\Gamma(c+1)}{\lambda^{c+1}} {}_3F_2 \left( \begin{matrix} b, & c-b, & \frac{c}{2}+1 \\ c, & \frac{c}{2} \end{matrix}; \frac{w^2}{\lambda^2} \right), \\ &(c > -1; \lambda > 0; |w| < \lambda), \end{aligned} \quad (24)$$

$$\begin{aligned} J_c^{10} \left( \frac{c}{2}, \frac{c}{2}; \lambda, w, -w \right) &= \int_0^\infty x^c e^{-\lambda x} {}_1F_1 \left( \frac{c}{2}; c; wx \right) {}_1F_1 \left( \frac{c}{2}; c; -wx \right) dx = \frac{\Gamma(c+1)}{\lambda^{c+1}} {}_2F_1 \left( \begin{matrix} \frac{c}{2}, & \frac{c}{2}+1 \\ c \end{matrix}; \frac{w^2}{\lambda^2} \right), \\ &(c > -1; \lambda > 0; |w| < \lambda). \end{aligned} \quad (25)$$

On other hand, by means of ([7], formula 7.2.4.68)

$$F_2(a; b, c-b; c, c; z, z) = (1-z)^{-a} {}_4F_3 \left( \frac{a}{2}, \frac{a+1}{2}, b, c-b; \frac{c}{2}, \frac{c+2}{2}, c; \frac{z^2}{(1-z)^2} \right), \quad (26)$$

it follows that

$$\begin{aligned} J_c^{j0}(b, c-b; \lambda, z, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; zx) {}_1F_1(c-b; c; zx) dx = \frac{\Gamma(c+j)}{(\lambda-z)^{c+j}} {}_4F_3 \left( b, c-b, \frac{c+j}{2}, \frac{c+j+1}{2}; \frac{z^2}{(\lambda-z)^2} \right), \\ &\quad (c+j > 0; \lambda > 0; c \neq 0, -1, -2, \dots, |w| < \lambda), \end{aligned} \quad (27)$$

$$J_c^{10}(b, c-b; \lambda, z, z) = \int_0^\infty x^c e^{-\lambda x} {}_1F_1(b; c; zx) {}_1F_1(c-b; c; zx) dx = \frac{\Gamma(c+1)}{(\lambda-z)^{c+1}} {}_3F_2 \left( b, c-b, \frac{c}{2}+1; \frac{z^2}{(\lambda-z)^2} \right), \quad (28)$$

$$(c > -1; \lambda > 0; |z| < |\lambda|),$$

$$\begin{aligned} J_c^{10} \left( \frac{c}{2}, \frac{c}{2}; \lambda, z, z \right) &= \int_0^\infty x^c e^{-\lambda x} \left[ {}_1F_1 \left( \frac{c}{2}; c; zx \right) \right]^2 dx = \frac{\Gamma(c+1)}{(\lambda-z)^{c+1}} {}_2F_1 \left( \frac{c}{2}, \frac{c}{2}+1; \frac{z^2}{(\lambda-z)^2} \right), \\ &\quad (c > -1; \lambda > 0; |z| < \lambda). \end{aligned} \quad (29)$$

By means of the identity ([6], Theorem 3, formula 29)

$$\begin{aligned} F_2(\sigma; \alpha_1, \alpha_2; \beta_1, \beta_2 + n; w, z) &= \frac{(\beta_2)_n}{(\beta_2 - \alpha_2)_n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(\alpha_2)_k}{(\beta_2)_k} F_2(\sigma; \alpha_1, \alpha_2 + k; \beta_1, \beta_2 + k; w, z) \\ &\quad (|w| + |z| < 1; n = 0, 1, 2, \dots; \beta_1, \beta_2 \neq 0, -1, -2, \dots; \beta_2 > \alpha_2), \end{aligned} \quad (30)$$

it easily follow

$$\begin{aligned} J_c^{jp}(b, b'; \lambda; w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c+p; zx) dx \\ &= \frac{\Gamma(c+j) (c)_p}{\lambda^{c+j} (c-b')_p} \sum_{k=0}^p \frac{(-p)_k (b')_k}{k! (c)_k} F_2 \left( c+j; b, b'+k; c, c+k; \frac{w}{\lambda}, \frac{z}{\lambda} \right), \\ &\quad (c+j > 0; p \geq 0; \lambda > 0; c \neq 0, -1, -2, \dots; \text{if } c-b' \text{ (negative integer), } b'-c \geq p \text{ } |w| + |z| < \lambda), \end{aligned} \quad (31)$$

Further by means of ([6], Theorem 3, formula 29)

$$\begin{aligned} F_2 \left( \sigma; \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2 - n}; w, z \right) &= \frac{1}{\left[ \prod_{i=0}^n (\beta_2 - i) \right]} \sum_{k=0}^n \binom{n}{k} \left[ \prod_{j=0}^{n-k} (\beta_2 - j) \right] \frac{(\sigma)_k (\alpha_2)_k}{(\beta_2)_k} z^k F_2 \left( \sigma+k; \frac{\alpha_1}{\beta_1}, \frac{\alpha_2+k}{\beta_2+k}; w, z \right), \\ &\quad (\beta_2 \neq i, i = 0, 1, \dots, n; \beta_1, \beta_2 - n \neq 0, -1, -2, \dots; n = 0, 1, 2, \dots; |w| + |z| < 1), \end{aligned} \quad (32)$$

it follows

$$\begin{aligned} J_c^{j(-p)}(b, b'; \lambda; w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c-p; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j}} \sum_{k=0}^p \frac{(-p)_k (b')_k (c+j)_k}{(c)_k (c-p)_k k!} \left( -\frac{z}{\lambda} \right)^k F_2 \left( c+k+j; b, b'+k; \frac{w}{\lambda}, \frac{z}{\lambda} \right), \\ &\quad (c+j > 0; p \geq 0; \lambda > 0; c \neq 0, -1, -2, \dots; c-p \neq 0, -1, -2, \dots; |w| + |z| < \lambda) \end{aligned} \quad (33)$$

whence

$$\begin{aligned} J_c^{j(-p)}(b, c; \lambda; w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(c; c-p; zx) dx \\ &= \frac{\Gamma(c+j) (\lambda-z)^{b-c-j}}{(\lambda-w-z)^b} \sum_{k=0}^p \frac{(-p)_k (c+j)_k}{(c-p)_k k!} \left( \frac{z}{z-\lambda} \right)^k {}_2F_1 \left( -k-j, b; \frac{w}{c; w+z-\lambda} \right) \\ &\quad (c+j > 0; p \geq 0; \lambda > 0; |w| + |z| < \lambda; c \neq 0, -1, -2, \dots; c-p \neq 0, -1, -2, \dots). \end{aligned} \quad (34)$$

The following identities are straightforward consequences of the pervious integrals:

$$\begin{aligned} J_c^{j0}(c+j, 0; \lambda, w, 0) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c+j; c; wx) dx = \frac{\Gamma(c+j)}{(\lambda-w)^{c+j}} {}_2F_1\left(-j, \begin{matrix} c+j \\ c \end{matrix}; \frac{w}{w-\lambda}\right), \\ &(c+j > 0; \lambda > 0; c \neq 0, -1, -2, \dots; |w| < \lambda), \end{aligned} \quad (35)$$

$$\begin{aligned} J_c^{j(\pm p)}(0, b; \lambda, 0, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c \pm p; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j}} {}_2F_1\left(c+j, \begin{matrix} b \\ c \pm p \end{matrix}; \frac{z}{\lambda}\right) = \frac{\Gamma(c+j)}{\lambda^{\pm p-b+c}(\lambda-z)^{b \mp p+j}} {}_2F_1\left(\pm p-j, \begin{matrix} c \pm p-b \\ c \pm p \end{matrix}; \frac{z}{\lambda}\right), \\ &(c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, -2, \dots; j = 0, \pm 1, \dots; p = 0, 1, 2, \dots; |z| < \lambda), \end{aligned} \quad (36)$$

$$\begin{aligned} J_c^{jj}(0, b; \lambda, 0, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c+j; zx) dx = \frac{\Gamma(c+j)}{\lambda^{c-b+j}(\lambda-z)^b}, \\ &(c+j > 0; \lambda > 0; j = 0, \pm 1, \dots; |z| < \lambda), \end{aligned} \quad (37)$$

$$\begin{aligned} J_c^{j0}(0, b; \lambda, 0, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; zx) dx = \frac{\Gamma(c+j)}{\lambda^{c+j-b}(\lambda-z)^b} \left[ 1 + \frac{bz}{c(\lambda-z)} \sum_{k=1}^j {}_2F_1\left(-j+k, \begin{matrix} b+1 \\ c+1 \end{matrix}; \frac{z}{\lambda-z}\right) \right], \\ &(c+j > 0; \lambda > 0; c \neq 0, -1, -2, \dots; |z| < \lambda). \end{aligned} \quad (38)$$

$$J_c^{00}(0, b; \lambda, 0, z) = \int_0^\infty x^{c-1} e^{-\lambda x} {}_1F_1(b; c; zx) dx = \frac{\lambda^{b-c} \Gamma(c)}{(\lambda-z)^b}, \quad (c > 0; \lambda > 0; |z| < \lambda). \quad (39)$$

### III. GORDON'S INTEGRAL AND CONFLUENT HYPERGEOMETRIC POLYNOMIALS

In the case of  $\alpha = -n$ , the confluent hypergeometric function  ${}_1F_1(\alpha; \beta; z)$  reduces to  $n$ -degree polynomial in  $z$ , namely  ${}_1F_1(-n; \beta; z) = \sum_{k=0}^n (-n)_k z^k / ((\beta)_k k!)$ ,  $n = 0, 1, \dots$ . Thus,

$$\begin{aligned} J_c^{j(\pm p)}(b, -n; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(-n; c \pm p; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j-b}(\lambda-w)^b} \sum_{k=0}^n \frac{(-n)_k (c+j)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1(-j-k, b; c; \frac{w}{w-\lambda}), \\ &(c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p = 0, 1, \dots; |w| < |\lambda|), \end{aligned} \quad (40)$$

where a direct differentiation of both sides with respect to  $z$  yields

$$\begin{aligned} J_c^{(j+m)(\pm p+m)}(b, m-n; \lambda, w, z) &= \int_0^\infty x^{c+j+m-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(m-n; c \pm p+m; zx) dx \\ &= \frac{(-1)^m \Gamma(c+j)(c \pm p)_m}{(-n)_m z^m \lambda^{c+j-b}(\lambda-w)^b} \sum_{k=m}^n \frac{(-k)_m (-n)_k (c+j)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1(-j-k, b; c; \frac{w}{w-\lambda}), \\ &(m \leq n; c+j+m > 0; \lambda > 0; c, c \pm p+m \neq 0, -1, \dots; |w| < |\lambda|). \end{aligned} \quad (41)$$

Further, setting  $b = -m$  in equation (40) implies

$$\begin{aligned} J_c^{jp}(-m, -n; \lambda; w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-m; c; wx) {}_1F_1(-n; c \pm p; zx) dx \\ &= \frac{\Gamma(c+j)}{\lambda^{c+j}} \sum_{k=0}^m \frac{(c+j)_k (-m)_k}{(c)_k k!} \left(\frac{w}{\lambda}\right)^k {}_2F_1(-n, c+j+k; c \pm p; \frac{z}{\lambda}) \\ &\equiv \frac{\Gamma(c+j)}{\lambda^{c+j}} \sum_{k=0}^n \frac{(c+j)_k (-n)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1(-m, c+j+k; c; \frac{w}{\lambda}), \\ &(c+j > 0, \lambda > 0; c, c \pm p \neq 0, -1, \dots; j = 0, \pm 1, \dots; n, m = 0, 1, \dots), \end{aligned} \quad (42)$$

where a direct differentiation of both sides with respect to  $w$  yields

$$\begin{aligned} \int_0^\infty x^{c+j+l-1} e^{-\lambda x} {}_1F_1(l-m; c+l; wx) {}_1F_1(-n; c \pm p; zx) dx \\ = \frac{(-1)^l \Gamma(c+j)(c)_l}{\lambda^{c+j} w^l (-m)_l} \sum_{k=l}^m \frac{(c+j)_k (-m)_k (-k)_l}{(c)_k k!} \left(\frac{w}{\lambda}\right)^k {}_2F_1(-n, c+j+k; c \pm p; \frac{z}{\lambda}), \\ (l \leq m; c+j+l > 0, \lambda > 0; c+l, c \pm p \neq 0, -1, \dots; j=0, \pm 1, \dots; n, m=0, 1, \dots), \end{aligned} \quad (43)$$

with a further differentiation of both sides with respect to  $z$  yields

$$\begin{aligned} \int_0^\infty x^{c+j+k+s-1} e^{-\lambda x} {}_1F_1(l-m; c+l; wx) {}_1F_1(s-n; c \pm p+s; zx) dx \\ = \frac{(-1)^l \Gamma(c+j)(c)_l}{\lambda^{c+j+s} w^l (-m)_l} \sum_{k=l}^m \frac{(c+j+k)_s (c+j)_k (-m)_k (-k)_l}{(c)_k k!} \left(\frac{w}{\lambda}\right)^k {}_2F_1(s-n, c+j+k+s; c \pm p+s; \frac{z}{\lambda}) \\ (s \leq n; l \leq m; c+j+l+s > 0, \lambda > 0; c+l, c \pm p+s \neq 0, -1, \dots; s, l, j=0, \pm 1, \dots; n, m=0, 1, \dots). \end{aligned} \quad (44)$$

If  $z = \lambda$ , equation (40) reads

$$\begin{aligned} J_c^{j(\pm p)}(-m, -n; \lambda; w, \lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-m; c; wx) {}_1F_1(-n; c \pm p; \lambda x) dx \\ &= \frac{\Gamma(c+j)(\pm p-j)_n}{\lambda^{c+j} (\pm p+c)_n} {}_3F_2(-m, c+j, 1+j \mp p; c, 1+j-n \mp p; \frac{w}{\lambda}), \\ (c+j > 0, \lambda > 0; c, c \pm p \neq 0, -1, \dots; j=0, \pm 1, \dots; n, m=0, 1, \dots; 1+j-n \pm p \neq 0, -1, \dots), \end{aligned} \quad (45)$$

and if  $w = \lambda$  equation (40) reads

$$\begin{aligned} J_c^{j(\pm p)}(-m, -n; \lambda; \lambda, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-m; c; \lambda x) {}_1F_1(-n; c \pm p; zx) dx \\ &= \frac{\Gamma(c+j)(-j)_m}{\lambda^{c+j} (c)_m} {}_3F_2(-n, c+j, 1+j; c \pm p, 1+j-m; \frac{z}{\lambda}), \\ (c+j > 0, \lambda > 0; c, c \pm p \neq 0, -1, \dots; j=0, \pm 1, \dots; n, m=0, 1, \dots; 1+j-m \neq 0, -1, \dots). \end{aligned} \quad (46)$$

Further if  $w = \lambda$ , equation (45) reads

$$\begin{aligned} J_c^{jp}(-m, -n; \lambda; \lambda, \lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-m; c; \lambda x) {}_1F_1(-n; c \pm p; \lambda x) dx \\ &= \frac{\Gamma(c+j)(\pm p-j)_n}{\lambda^{c+j} (\pm p+c)_n} {}_3F_2(-m, c+j, 1+j \mp p; c, 1+j-n \mp p; 1) \\ &= \frac{\Gamma(c+j)(-j)_m}{\lambda^{c+j} (c)_m} {}_3F_2(-n, c+j, 1+j; c \pm p, 1+j-m; 1), \\ (c+j > 0, \lambda > 0; c, c \pm p \neq 0, -1, \dots; j=0, \pm 1, \dots; n, m=0, 1, \dots; 1+j-n \pm p, 1+j-m \neq 0, -1, \dots), \end{aligned} \quad (47)$$

whence

$$\begin{aligned} J_c^{j0}(-m, -n; \lambda; \lambda, \lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-m; c; \lambda x) {}_1F_1(-n; c; \lambda x) dx \\ &= \frac{\Gamma(c+j)(-j)_n}{\lambda^{c+j} (c)_n} {}_3F_2(-m, c+j, 1+j; c, 1+j-n; 1), \\ &\equiv \frac{\Gamma(c+j)(-j)_m}{\lambda^{c+j} (c)_m} {}_3F_2(-n, c+j, 1+j; c, 1+j-m; 1), \\ (c+j > 0, \lambda > 0; c \neq 0, -1, \dots; j=0, \pm 1, \dots; n, m=0, 1, \dots; 1+j-n, 1+j-m \neq 0, -1, \dots). \end{aligned} \quad (48)$$

If  $j = 0$ , equation (47)

$$\begin{aligned} J_c^{0p\pm}(-n, -m; \lambda, \lambda, \lambda) &= \int_0^\infty x^{c-1} e^{-\lambda x} {}_1F_1(-n; c; \lambda x) {}_1F_1(-m; c \pm p; \lambda x) dx = \frac{\Gamma(c)}{\lambda^c} \frac{m!}{(m-n)!} \frac{(\pm p)_{m-n}}{(c \pm p)_m}, \\ (m \geq n; c > 0, \lambda > 0; c \pm p \neq 0, -1, \dots; j=0, \pm 1, \dots; n, m=0, 1, \dots). \end{aligned} \quad (49)$$

If  $m = n$ , equation (47)

$$J_c^{j0}(-n, -n; \lambda; \lambda, \lambda) = \int_0^\infty x^{c+j-1} e^{-\lambda x} [{}_1F_1(-n; c; \lambda x)]^2 dx = \frac{\Gamma(c+j) (-j)_n}{\lambda^{c+j} (c)_n} {}_3F_2(-n, c+j, 1+j; c, 1+j-n; 1),$$

$$(c+j > 0, \lambda > 0; c \neq 0, -1, \dots; j = 0, \pm 1, \dots; n = 0, 1, \dots; 1+j-n \neq 0, -1, \dots). \quad (50)$$

The condition  $1+j-n \neq 0, -1, -2, \dots$  in (50) can be softened using the identity

$$(-j)_n {}_3F_2(-n, c+j, 1+j; c, 1+j-n; 1) = n! {}_3F_2(-n, -j, j+1; c, 1; 1), \quad (51)$$

to yield

$$J_c^{j0}(-n, -n; \lambda; \lambda, \lambda) = \int_0^\infty x^{c+j-1} e^{-\lambda x} [{}_1F_1(-n; c; \lambda x)]^2 dx = \frac{\Gamma(c+j) n!}{\lambda^{c+j} (c)_n} {}_3F_2(-n, -j, 1+j; c, 1; 1),$$

$$(c+j > 0; \lambda > 0; c \neq 0, -1, -2, \dots; j = 0, \pm 1, \pm 2, \dots; n = 0, 1, 2, \dots). \quad (52)$$

and thus

$$J_c^{10}(-n, -n; \lambda; \lambda, \lambda) = \int_0^\infty x^c e^{-\lambda x} [{}_1F_1(-n; c; \lambda x)]^2 dx = \frac{\Gamma(c) n!}{\lambda^{c+1} (c)_n} (c+2n), \quad (c > 0; \lambda > 0). \quad (53)$$

From the symmetric property  $j \longleftrightarrow -j-1$  of  ${}_3F_2(-n, -j, 1+j; c, 1; 1)$ , it also follow

$$J_c^{(-j-1)0}(-n, -n; \lambda; \lambda, \lambda) = \int_0^\infty x^{c-j-2} e^{-\lambda x} [{}_1F_1(-n; c; \lambda x)]^2 dx = \frac{\Gamma(c-j-1) n!}{\lambda^{c-j-1} (c)_n} {}_3F_2(-n, -j, 1+j; c, 1; 1),$$

$$(c-j-2 > 0; \lambda > 0; c \neq 0, -1, -2, \dots; j = 0, \pm 1, \pm 2, \dots; n = 0, 1, 2, \dots) \quad (54)$$

whence

$$J_c^{(-j-1)0}(-n, -n; \lambda; \lambda, \lambda) = \frac{\Gamma(c-j-1) \lambda^{2j+1}}{\Gamma(c+j)} J_c^{j0}(-n, -n; \lambda; \lambda, \lambda), \quad (55)$$

for example

$$J_c^{(-2)0}(-n, -n; \lambda; \lambda, \lambda) = \frac{\Gamma(c-2) \lambda^3}{\Gamma(c+1)} J_c^{10}(-n, -n; \lambda; \lambda, \lambda) = \frac{\Gamma(c-2) n!}{c \lambda^{c-2} (c)_n} (c+2n). \quad (56)$$

Further recurrence relations of this type are developed in the appendix. Note, from equations (42) and (45), it follows

$$\sum_{k=0}^n \frac{(-n)_k (c+j)_k}{(c \pm p)_k k!} {}_2F_1\left(-m, c+j+k; c; \frac{w}{\lambda}\right) = \frac{(\pm p-j)_n}{(\pm p+c)_n} {}_3F_2\left(-m, c+j, 1+j \mp p; c, 1+j-n \mp p; \frac{w}{\lambda}\right). \quad (57)$$

An important class of W. Gordon's integral occur in the case of  $w = k_1, z = k_2$ , and  $\lambda = (k_1 + k_2)/2$ , namely,

$$J_c^{j(\pm p)}\left(-n, -m; \frac{k_1+k_2}{2}, k_1, k_2\right) = \int_0^\infty x^{c+j-1} e^{-(k_1+k_2)x/2} {}_1F_1(-n; c; k_1 x) {}_1F_1(-m; c \pm p; k_2 x) dx$$

$$= \frac{2^{c+j} \Gamma(c+j)}{(k_1+k_2)^{c+j}} F_2\left(c+j; \begin{matrix} -n, & -m \\ c, & c \pm p \end{matrix}; \frac{2k_1}{k_1+k_2}, \frac{2k_2}{k_1+k_2}\right),$$

$$(k_1+k_2 > 0; c+j > 0; c, c \pm p \neq 0, \pm 1, \dots) \quad (58)$$

equivalently,

$$J_c^{j(\pm p)}\left(-n, -m; \frac{k_1+k_2}{2}, k_1, k_2\right) = \int_0^\infty x^{c+j-1} e^{-(k_1+k_2)x/2} {}_1F_1(-n; c; k_1 x) {}_1F_1(-m; c \pm p; k_2 x) dx$$

$$= \begin{cases} \frac{\Gamma(c+j)(\pm p-j)_m}{k_1^{c+j} (\pm p+c)_m} {}_3F_2(-n, c+j, 1+j \mp p; c, 1+j-m \mp p; 1), & \text{if } k_1 = k_2, \\ \frac{2^{c+j} \Gamma(c+j)}{(k_1+k_2)^{c+j}} \left(\frac{k_1-k_2}{k_1+k_2}\right)^m \sum_{i=0}^{\min\{j \mp p, m\}} \frac{(-j \pm p)_i (-m)_i}{(c \pm p)_i i!} \left(\frac{2k_2}{k_2-k_1}\right)^i \\ \times F_1\left(-n, c+j+m, i-m; c; \frac{2k_1}{k_1+k_2}, \frac{2k_1}{k_1-k_2}\right), & \text{if } k_1 \neq k_2. \end{cases} \quad (59)$$



and further equivalent to

$$\begin{aligned}
J_c^{j(\pm p)} \left( -n, -m; \frac{k_1 + k_2}{2}, k_1, k_2 \right) &= \int_0^\infty x^{c+j-1} e^{-(k_1+k_2)x/2} {}_1F_1(-n; c; k_1 x) {}_1F_1(-m; c \pm p; k_2 x) dx \\
&= \begin{cases} \frac{\Gamma(c+j)(\pm p-j)_m}{k_1^{c+j}(\pm p+c)_m} {}_3F_2(-n, c+j, 1+j \mp p; c, 1+j-m \mp p; 1), & \text{if } k_1 = k_2, \\ \frac{(-1)^n 2^{c+j} \Gamma(c+j)}{(k_1+k_2)^{c+j}} \left( \frac{k_1-k_2}{k_1+k_2} \right)^{m+n} \sum_{i=0}^{\min\{j \mp p, m\}} \frac{(-j \pm p)_i (-m)_i}{(c \pm p)_i i!} \left( \frac{2k_2}{k_2-k_1} \right)^i \\ \times \sum_{r=0}^{j+i} \frac{(-n)_r (-j-i)_r}{(c)_r r!} \left( \frac{2k_2}{k_2-k_1} \right)^r {}_2F_1 \left( r-n, i-m; c+r; \frac{-4k_1 k_2}{(k_1-k_2)^2} \right), & \text{if } k_1 \neq k_2. \end{cases} \quad (60)
\end{aligned}$$

In particular

$$\begin{aligned}
J_c^{j0} \left( -n, -m; \frac{k_1 + k_2}{2}, k_1, k_2 \right) &= \int_0^\infty x^{c+j-1} e^{-(k_1+k_2)x/2} {}_1F_1(-n; c; k_1 x) {}_1F_1(-m; c; k_2 x) dx \\
&= \begin{cases} \frac{\Gamma(c+j)(-j)_m}{k_1^{c+j}(\pm p+c)_m} {}_3F_2(-n, c+j, 1+j; c, 1+j-m; 1), & (k_1 = k_2, m \leq j), \\ \frac{(-1)^n 2^{c+j} \Gamma(c+j)}{(k_1+k_2)^{c+j}} \left( \frac{k_1-k_2}{k_1+k_2} \right)^{m+n} \sum_{i=0}^m \frac{(-j)_i (-m)_i}{(c)_i i!} \left( \frac{2k_2}{k_2-k_1} \right)^i \\ \times \sum_{r=0}^{j+i} \frac{(-n)_r (-j-i)_r}{(c)_r r!} \left( \frac{2k_2}{k_2-k_1} \right)^r {}_2F_1 \left( r-n, i-m; c+r; \frac{-4k_1 k_2}{(k_1-k_2)^2} \right), & (k_1 \neq k_2, m \leq j), \end{cases} \quad (61)
\end{aligned}$$

and

$$\begin{aligned}
J_c^{00} \left( -m, -n; \frac{k_1 + k_2}{2}; k_1, k_2 \right) &= \int_0^\infty x^{c-1} e^{-(k_1+k_2)x/2} {}_1F_1(-m; c; k_1 x) {}_1F_1(-n; c; k_2 x) dx \\
&= \frac{2^c \Gamma(c)}{(k_1+k_2)^c} \sum_{k=0}^m \frac{(-m)_k}{k!} \left( \frac{2k_1}{k_1+k_2} \right)^k {}_2F_1(-n, c+k; c; \frac{2k_2}{k_1+k_2}), \\
&(c > 0, \lambda > 0; c \neq 0, -1, -2, \dots; n, m = 0, 1, 2, \dots), \quad (62)
\end{aligned}$$

where, generally,

$$\begin{aligned}
J_c^{00}(-m, -n; \lambda; w, z) &= \int_0^\infty x^{c-1} e^{-\lambda x} {}_1F_1(-m; c; w x) {}_1F_1(-n; c; z x) dx \\
&= \frac{\Gamma(c)}{\lambda^c} \sum_{k=0}^m \frac{(-m)_k}{k!} \left( \frac{w}{\lambda} \right)^k {}_2F_1(-n, c+k; c; \frac{z}{\lambda}), \\
&(c > 0, \lambda > 0; c \neq 0, -1, -2, \dots; n, m = 0, 1, 2, \dots), \quad (63)
\end{aligned}$$

from which the classical orthogonality property of the confluent hypergeometric functions follows, namely,

$$\begin{aligned}
J_c^{00}(-m, -n; \lambda; \lambda, \lambda) &= \int_0^\infty x^{c-1} e^{-\lambda x} {}_1F_1(-m; c; \lambda x) {}_1F_1(-n; c; \lambda x) dx = \frac{\Gamma(c) n!}{\lambda^c (c)_n} \delta_{nm}, \\
&(c > 0, \lambda > 0; c \neq 0, -1, -2, \dots; \delta_{nm} = 0 \text{ if } n \neq m, \delta_{nm} = 1 \text{ if } n = m), \quad (64)
\end{aligned}$$

using  $\sum_{k=0}^m (-m)_k (-k)_n / k! = n! \delta_{nm}$ . The same conclusion also follows from equation (48) using the fact that

$$\lim_{j \rightarrow 0} (-j)_m {}_3F_2(-n, c+j, 1+j; c, 1+j-m; 1) = n! \delta_{nm}.$$

If in equation (1),  $b = b' = -n$  and  $p = 0$ , it follows

$$\begin{aligned}
J_c^{j0}(-n, -n; \lambda; w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-n; c; w x) {}_1F_1(-n; c; z x) dx \\
&= \frac{n! \Gamma(c+j)}{\lambda^{c+j} (c)_n} \sum_{k=0}^n \frac{(c+j)_k (-n)_k}{(c)_k k!} \left( \frac{z}{\lambda} \right)^k P_n^{(c-1, j+k-n)} \left( 1 - \frac{2w}{\lambda} \right), \quad (65)
\end{aligned}$$

where  $P_n^{(\alpha, \beta)}(z)$  is the Jacobi polynomial of order  $\alpha$ ,  $\beta$  and degree  $n$  in  $z$ . The relation  $P_n^{(a, b)}(-1) = (-1)^n (b+1)_n / n!$  reduce the equation (65) to

$$\begin{aligned} J_c^{j0}(-n, -n; \lambda; \lambda, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-n; c; \lambda x) {}_1F_1(-n; c; z x) dx \\ &= \frac{\Gamma(c+j)(-j)_n}{\lambda^{c+j} (c)_n} {}_3F_2\left(-n, c+j, 1+j; c, 1+j-n; \frac{z}{\lambda}\right), \\ &\quad (c+j > 0, \lambda > 0; c \neq 0, -1, \dots; n = 0, 1, \dots; 1+j-n \neq 0, -1, \dots), \end{aligned} \quad (66)$$

as expected. From equation (66), it follows

$$\begin{aligned} J_c^{n0}(-n, -n; \lambda; \lambda, z) &= \int_0^\infty x^{c+n-1} e^{-\lambda x} {}_1F_1(-n; c; \lambda x) {}_1F_1(-n; c; z x) dx = \frac{(-1)^n \Gamma(c) n!}{\lambda^{c+n}} {}_3F_2\left(-n, c+n, 1+n; c, 1; \frac{z}{\lambda}\right), \\ &\quad (c+n > 0, \lambda > 0; c \neq 0, -1, \dots). \end{aligned} \quad (67)$$

For  $n \geq m$

$$\begin{aligned} J_c^{(n-m)(\pm p)}(-n, -m; \lambda, \lambda, z) &= \int_0^\infty x^{c+n-m-1} e^{-\lambda x} {}_1F_1(-n; c; \lambda x) {}_1F_1(-m; c \pm p; z x) dx \\ &= \frac{(-1)^{m+n} \Gamma(c) n!}{\lambda^{c+n-m} (c \pm p)_m} \left(\frac{z}{\lambda}\right)^m, \quad (c+n-m > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p \geq 0). \end{aligned} \quad (68)$$

The following integral follows immediately

$$\begin{aligned} J_c^{j(\pm p)}(0, -n; \lambda, 0, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-n; c \pm p; z x) dx = \frac{\Gamma(c+j)}{\lambda^{c+j}} {}_2F_1\left(-n, c+j; c \pm p; \frac{z}{\lambda}\right), \\ &\quad (c+j > 0; \lambda > 0; c \pm p \neq 0, -1, \dots; p = 0, 1, \dots) \end{aligned} \quad (69)$$

whence

$$\begin{aligned} J_c^{j(\pm p)}(0, -n; \lambda, 0, \lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-n; c \pm p; \lambda x) dx = \begin{cases} \frac{\Gamma(c+j)}{\lambda^{c+j}} \frac{(\pm p - j)_n}{(c \pm p)_n}, & \text{if } j \mp p \geq n, \\ 0, & \text{if } j \mp p < n, \end{cases} \\ &\quad (c+j > 0; \lambda > 0; c \pm p \neq 0, -1, \dots; p = 0, 1, \dots), \end{aligned} \quad (70)$$

and if  $p = j$ ,

$$J_c^{jj}(0, -n; \lambda, 0, z) = \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(-n; c+j; z x) dx = \frac{\Gamma(c+j)}{\lambda^{c+j}} \left(1 - \frac{z}{\lambda}\right)^n, \quad (c+j > 0; \lambda > 0). \quad (71)$$

and

$$J_c^{j0}(0, -n; \lambda, 0, \lambda) = \int_0^\infty x^{c+n-1} e^{-\lambda x} {}_1F_1(-n; c; \lambda x) dx = \frac{(-1)^n n! \Gamma(c)}{\lambda^{c+n}}, \quad (c+n > 0; \lambda > 0; c \neq 0, -1, \dots). \quad (72)$$

#### IV. GORON'S INTEGRAL AND SPECIAL FUNCTIONS

The generalized Laguerre polynomials are defined, for integer  $n$ , in terms of confluent hypergeometric functions by

$$L_n^\lambda(z) = \frac{(\lambda+1)_n}{n!} {}_1F_1(-n; \lambda+1; z), \quad (73)$$

thus,

$$\begin{aligned} \int_0^\infty x^{c+j-1} e^{-\lambda x} L_n^{c \pm p-1}(z x) {}_1F_1(b; c; w x) dx &= \frac{\Gamma(c+j)(c \pm p)_n}{n! \lambda^{c+j}} \sum_{k=0}^n \frac{(-n)_k (c+j)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1\left(c+j+k, b; c; \frac{w}{\lambda}\right) \\ &\quad (c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p = 0, 1, \dots; |w| < |\lambda|), \end{aligned} \quad (74)$$

whence, if  $b = 0$ ,

$$\int_0^\infty x^{c+j-1} e^{-\lambda x} L_n^{c \pm p-1}(z x) dx = \begin{cases} \frac{\Gamma(c+j)(c \pm p)_n}{n! \lambda^{c+j}} {}_2F_1\left(-n, c+j; c \pm p; \frac{z}{\lambda}\right), & \text{if } z \neq \lambda, c \pm p \neq 0, -1, \dots, \\ \frac{\Gamma(c+j)(\pm p-j)_n}{\lambda^{c+j} n!}, & \text{if } z = \lambda, j \mp p \geq n, \end{cases} \quad (c+j > 0; \lambda > 0; p = 0, 1, \dots). \quad (75)$$

From equation (75), it follows

$$\begin{aligned} \int_0^\infty x^c e^{-\lambda x} L_n^c(\lambda x) dx &= \begin{cases} \frac{\Gamma(c+1)}{\lambda^{1+c}}, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1; c > -1, \lambda > 0, n = 0, 1, \dots, \end{cases} \\ \int_0^\infty x^{c+j} e^{-\lambda x} L_n^c(\lambda x) dx &= \begin{cases} \frac{\Gamma(c+j+1)(-j)_n}{\lambda^{j+c+1} n!}, & \text{if } n < j, \\ (-1)^n \frac{\Gamma(c+n+1)}{\lambda^{n+c+1}}, & \text{if } n = j, \\ 0, & \text{if } n > j; c+j > -1, \lambda > 0, n = 0, 1, \dots, \end{cases} \\ \int_0^\infty x^c e^{-\lambda x} L_n^{c-p}(\lambda x) dx &= \begin{cases} \frac{\Gamma(c+1)(-p)_n}{\lambda^{c+1} n!}, & \text{if } n < p, \\ (-1)^n \frac{\Gamma(c+1)}{\lambda^{c+1}}, & \text{if } n = p, \\ 0, & \text{if } n > p; c > -1, \lambda > 0, c-p \geq 0. \end{cases} \end{aligned} \quad (76)$$

Since

$$\frac{d^m}{dz^m} L_n^\lambda(az) = (-a)^m L_{n-m}^{\lambda+m}(az)$$

it easily follows, for  $n \geq m$  and  $m = 0, 1, 2, \dots$ , that

$$\begin{aligned} \int_0^\infty x^{c+j+m-1} e^{-\lambda x} L_{n-m}^{c \pm p+m-1}(z x) {}_1F_1(b; c; w x) dx \\ = \frac{(c \pm p)_n \Gamma(c+j)}{n! z^m \lambda^{c+j}} \sum_{k=m}^n \frac{(-k)_m (-n)_k (c+j)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1(c+j+k, b; c; \frac{w}{\lambda}). \end{aligned} \quad (77)$$

and, for  $\mu = 0, 1, 2, \dots, m \leq n$ ,

$$\begin{aligned} \int_0^\infty x^{c+j+m+\mu-1} e^{-\lambda x} L_{n-m}^{c \pm p+m-1}(z x) {}_1F_1(b+\mu; c+\mu; w x) dx \\ = \frac{\Gamma(c+j)(c+p)_n}{n! z^m \lambda^{c+j+\mu}} \sum_{k=0}^n \frac{(-k)_m (-n)_k (c+j)_k (c+j+k)_\mu}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1(c+j+k+\mu, b+\mu; c+\mu; \frac{w}{\lambda}). \end{aligned} \quad (78)$$

On other hand,

$$\begin{aligned} \int_0^\infty x^{c+j-1} e^{-\lambda x} L_n^{c \pm p-1}(z x) L_m^{c-1}(w x) dx = \frac{(c)_m (c \pm p)_n \Gamma(c+j)}{m! n! \lambda^{c+j}} \sum_{k=0}^n \frac{(-n)_k (c+j)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1\left(c+j+k, -m; c; \frac{w}{\lambda}\right), \\ (c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p = 0, 1, \dots). \end{aligned} \quad (79)$$

and by direct differentiation  $s$ -times, with respect to  $w$ , of both sides

$$\begin{aligned} & \int_0^\infty x^{c+j+s-1} e^{-\lambda x} L_n^{c \pm p-1}(zx) L_{m-s}^{c+s-1}(wx) dx \\ &= \frac{(-1)^s (c)_m (c \pm p)_n \Gamma(c+j)}{m! n! \lambda^{c+j+s}} \sum_{k=0}^n \frac{(-n)_k (c+j)_k}{(c \pm p)_k k!} \frac{(c+j+k)_s (-m)_s}{(c)_s} \left(\frac{z}{\lambda}\right)^k {}_2F_1\left(c+j+k+s, s-m; c+s; \frac{w}{\lambda}\right), \\ & (m \geq s; c+j+s > 0; \lambda > 0; c+s, c \pm p \neq 0, -1, \dots; p = 0, 1, \dots). \end{aligned} \quad (80)$$

and further differentiation of both sides  $\mu$ -times, with respect to  $z$ ,

$$\begin{aligned} & \int_0^\infty x^{c+j+s+\mu-1} e^{-\lambda x} L_{n-\mu}^{c \pm p+\mu-1}(zx) L_{m-s}^{c+s-1}(wx) dx \\ &= \frac{(-m)_s (c)_m (c \pm p)_n \Gamma(c+j)}{(-1)^s m! n! z^\mu \lambda^{c+j+s} (c)_s} \sum_{k=\mu}^n \frac{(-k)_\mu (c+j+k)_s (-n)_k (c+j)_k}{(c \pm p)_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1(c+j+k+s, s-m; c+s; \frac{w}{\lambda}), \\ & (s \leq m; \mu \leq n; c+j+s+\mu > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p = 0, 1, \dots). \end{aligned} \quad (81)$$

If  $w = \lambda$ , equation (79) reads

$$\begin{aligned} \int_0^\infty x^{c+j-1} e^{-\lambda x} L_n^{c \pm p-1}(zx) L_m^{c-1}(\lambda x) dx &= \frac{(-j)_m (c \pm p)_n \Gamma(c+j)}{m! n! \lambda^{c+j}} {}_3F_2\left(-n, c+j, 1+j; c \pm p, 1+j-m; \frac{z}{\lambda}\right), \\ & (j \geq m; c+j > 0; \lambda > 0; c, c \pm p \neq 0, -1, \dots; p = 0, 1, \dots). \end{aligned} \quad (82)$$

and if  $p = 0$  it yields

$$\begin{aligned} \int_0^\infty x^{c+j-1} e^{-\lambda x} L_n^{c+j-1}(zx) L_m^{c-1}(\lambda x) dx &= \frac{(-j)_m \Gamma(c+j+n)}{m! n! \lambda^{c+j}} {}_2F_1\left(-n, 1+j; 1+j-m; \frac{z}{\lambda}\right), \\ & (j \geq m; c+j > 0; \lambda > 0). \end{aligned} \quad (83)$$

and by taken limit of both sides as  $j \rightarrow 0$

$$\int_0^\infty x^{c-1} e^{-\lambda x} L_n^{c-1}(zx) L_m^{c-1}(\lambda x) dx = \frac{(-1)^m z^m \Gamma(c+n) (-n)_m}{m! n! \lambda^{c+n} (\lambda - z)^{m-n}}, \quad (n \geq m; c > 0; \lambda > 0; |z| < \lambda), \quad (84)$$

thus

$$\int_0^\infty x^{c-1} e^{-\lambda x} L_n^{c-1}(\lambda x) L_m^{c-1}(\lambda x) dx = \frac{(c)_n \Gamma(c)}{m! \lambda^c} \delta_{m,n}, \quad (c > 0; \lambda > 0; n, m = 0, 1, \dots). \quad (85)$$

By means of  $H_{2n}(\sqrt{z}) = (-1)^n (2n)! {}_1F_1(-n; 0.5; z)/n!$ , it follows using (74) that

$$\begin{aligned} \int_0^\infty x^{j-\frac{1}{2}} e^{-\lambda x} L_n^{\pm p-\frac{1}{2}}(zx) H_{2n}(\sqrt{wx}) dx &= \frac{(-1)^n (2n)! (\pm p + \frac{1}{2})_n \Gamma(j + \frac{1}{2})}{(n!)^2 \lambda^{j+\frac{1}{2}}} \\ &\times \sum_{k=0}^n \frac{(-n)_k (j + \frac{1}{2})_k}{(\pm p + \frac{1}{2})_k k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1\left(j+k+\frac{1}{2}, -n; \frac{1}{2}; \frac{w}{\lambda}\right), \quad (j > 1/2; p, n = 0, 1, \dots), \end{aligned} \quad (86)$$

from which it follows

$$\begin{aligned} \int_0^\infty x^{j-\frac{1}{2}} e^{-\lambda x} L_n^{\pm p-\frac{1}{2}}(zx) H_{2n}(\sqrt{\lambda x}) dx &= \frac{(-1)^n (2n)! (-j)_n (\pm p + \frac{1}{2})_n \Gamma(j + \frac{1}{2})}{(n!)^2 (\frac{1}{2})_n \lambda^{j+\frac{1}{2}}} \\ &\times {}_3F_2\left(j + \frac{1}{2}, 1+j, -n; 1+j-n, \pm p + \frac{1}{2}; \frac{z}{\lambda}\right), \quad (j > 1/2; \lambda > 0; p, n = 0, 1, \dots) \end{aligned} \quad (87)$$

However, by means of

$$\lim_{j \rightarrow 0} (-j)_n {}_3F_2\left(-n, \frac{1}{2} + j, j+1; \pm p + \frac{1}{2}, j+1-n; 1\right) = \frac{(2n)!}{4^n (\pm p + \frac{1}{2})_n} \quad (88)$$

it easily follows that

$$\int_0^\infty x^{-\frac{1}{2}} e^{-\lambda x} L_n^{\pm p - \frac{1}{2}}(\lambda x) H_{2n}(\sqrt{\lambda x}) dx = \frac{(-1)^n ((2n)!)^2 \sqrt{\pi}}{4^n (n!)^2 (\frac{1}{2})_n \sqrt{\lambda}}, \quad (\lambda > 0, p : \text{arbitrary}). \quad (89)$$

Note also, if  $c = 1/2$  and  $p = 1$ , it easily follows

$$\begin{aligned} \int_0^\infty x^{j-1} e^{-\lambda x} H_{2m}(\sqrt{wx}) H_{2n+1}(\sqrt{zx}) dx \\ = (-1)^{m+n} \frac{(2m)!}{m!} \frac{2\sqrt{z}(2n+1)!}{n!} \frac{\Gamma(\frac{1}{2}+j)}{\lambda^{\frac{1}{2}+j}} \sum_{k=0}^m \frac{(\frac{1}{2}+j)_k (-m)_k}{(\frac{1}{2})_k k!} \left(\frac{w}{\lambda}\right)^k {}_2F_1\left(-n, \frac{1}{2}+j+k; \frac{3}{2}; \frac{z}{\lambda}\right), \\ (j > 0; \lambda > 0; m, n = 0, 1, \dots), \end{aligned} \quad (90)$$

For  $j > n$  and  $z = \lambda$

$$\int_0^\infty x^{j-1} e^{-\lambda x} H_{2m}(\sqrt{wx}) H_{2n+1}(\sqrt{\lambda x}) dx = (-1)^{m+n} \frac{(2m)!}{m!} \frac{2(2n+1)!}{n!} \frac{\Gamma(\frac{1}{2}+j)}{\lambda^j} \frac{(1-j)_n}{(\frac{3}{2})_n} {}_3F_2\left(j + \frac{1}{2}, j, -m; \frac{1}{2}, j-n; \frac{w}{\lambda}\right) \quad (91)$$

and for  $c = 1/2$  and  $p = 0$

$$\int_0^\infty x^{j-\frac{1}{2}} e^{-\lambda x} H_{2m}(\sqrt{wx}) H_{2n}(\sqrt{zx}) dx = (-1)^{m+n} \frac{(2m)!}{m!} \frac{(2n)!}{n!} \frac{\Gamma(\frac{1}{2}+j)}{\lambda^{\frac{1}{2}+j}} \sum_{k=0}^m \frac{(\frac{1}{2}+j)_k (-m)_k}{(\frac{1}{2})_k k!} \left(\frac{w}{\lambda}\right)^k {}_2F_1\left(-n, \frac{1}{2}+j+k; \frac{1}{2}; \frac{z}{\lambda}\right) \quad (92)$$

We may remark that all the above results involving  ${}_1F_1$  can be rewritten in the representation using the Whittaker function because of the following relationship:

$${}_1F_1(a, b, z) = e^{z/2} z^{-b/2} M_{(b-2a)/2, (b-1)/2}(z)$$

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## V. APPENDIX

In this appendix we summarize some recurrence relations of the Gordon's integral that follow using the contiguous relations of the confluent hypergeometric functions:

$$J_{c+1}^{j(\pm p)}(b+1, b'; \lambda, w, z) = \frac{c}{w} \left[ J_c^{j(\pm p)}(b+1, b'; \lambda, w, z) - J_c^{j(\pm p)}(b, b'; \lambda, w, z) \right]. \quad (93)$$

$$\begin{aligned} J_c^{j(\pm p)}(b+1, b'; \lambda, w, z) &= \frac{c-b}{b} J_c^{j(\pm p)}(b-1, b'; \lambda, w, z) + \frac{w}{b} J_c^{(j+1)(\pm p)}(b, b'; \lambda, w, z) \\ &+ \frac{2b-c}{b} J_c^{j(\pm p)}(b, b'; \lambda, w, z), \end{aligned} \quad (94)$$

$$J_{c+1}^{j(\pm p)}(b+1, b'; \lambda, w, z) = \frac{(c+1-k)_k}{w^k} \sum_{m=0}^k \frac{(-1)^m k!}{m! (k-m)!} J_{c+1-k}^{j(\pm p+k)}(b+1-m, b'; \lambda, w, z) \quad (95)$$

$$J_c^{j(\pm p)}(b, b'; \lambda, w, z) = \frac{b}{c} J_{c+1}^{(j-1)(\pm p-1)}(b+1, b'; \lambda, w, z) - \frac{b-c}{c} J_{c+1}^{(j-1)(\pm p-1)}(b, b'; \lambda, w, z), \quad (96)$$

$$\begin{aligned} J_c^{j(\pm p)}(b+1, b'; \lambda, w, z) &= \frac{b}{c} J_{c+1}^{(j+1)(\pm p-1)}(b+1, b'; \lambda, w, z) + \frac{w}{c} J_{c+1}^{j(\pm p-1)}(b+1, b'; \lambda, w, z) \\ &- \frac{b-c}{c} J_{c+1}^{(j+1)(\pm p-1)}(b, b'; \lambda, w, z), \end{aligned} \quad (97)$$

$$J_c^{j(\pm p)}(b+1, b'; \lambda, w, z) = J_c^{j(\pm p)}(b, b'; \lambda, w, z) + \frac{w}{b} J_c^{(j+1)(\pm p)}(b, b'; \lambda, w, z) - \frac{w(b-c)}{cb} J_{c+1}^{j(\pm p-1)}(b, b'; \lambda, w, z), \quad (98)$$

$$J_{c-1}^{j(\pm p)}(b, b'; \lambda, w, z) = J_c^{(j-1)(\pm p)}(b, b'; \lambda, w, z) + \frac{w}{c-1} J_c^{j(\pm p-1)}(b, b'; \lambda, w, z) + \frac{w(b-c)}{c(1-c)} J_{c+1}^{(j-1)(\pm p-2)}(b, b'; \lambda, w, z), \quad (99)$$

$$J_c^{j0}(b, b'; \lambda, w, z) = \frac{b1-c}{z} J_c^{(j-1)0}(b, b'+1; \lambda, w, z) + \frac{b'-c}{z} J_c^{(j-1)0}(b, b'-1; \lambda, w, z) + \frac{c-2b'}{z} J_c^{(j-1)0}(b, b'; \lambda, w, z). \quad (100)$$